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Representations of the q-deformed algebra $U'_q(\mathrm{so}_4)$

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Abstract. Irreducible finite-dimensional representations of the q-deformed algebra $U'_q(\mathrm{so}_4)$, which is a generalization of Fairlie's algebra $U'_q(\mathrm{so}_3)$, are studied. These representations T_{rs} are given by two integral or half-integral numbers r and s such that $r \geqslant |s| \geqslant 0$. Spectra and eigenvectors of the operator $T_{rs}(I_{43})$ are given, where I_{43} is one of the generators of $U'_q(\mathrm{so}_4)$. By means of this result, explicit expressions for all generating operators $T_{rs}(I_{i,i-1})$, $i = 2, 3, 4$, with respect to the basis $|x, m\rangle$ corresponding to restriction onto the subalgebra $U'_q(\mathrm{so}_2) + U'_q(\mathrm{so}_2)$ are evaluated. By analytical continuation, infinite-dimensional representations of the 'non-compact' q-deformed algebra $U'_q(\mathrm{so}_{2,2})$ are found. They are characterized by two complex numbers.

1. Introduction

Quantum groups and algebras are of great importance for applications in quantum integrable systems, in quantum field theory and in statistical physics. There are many results on the applications of the simplest quantum groups for phenomenological descriptions in particle theory (Gavrilik 1994), in nuclear physics (Raychev *et al* 1990) etc. Quantum groups and algebras are applied mainly by means of their representations. Therefore, it is necessary to have a well developed theory of their representations.

In the classical case, the embedding $SO(3) \subset U(3)$ (and the more general embedding $SO(n) \subset U(n)$) is of great importance for a group-theoretical approach to some physical problems. In the framework of Drinfeld–Jimbo quantum groups, we cannot construct the corresponding embeddings. In this framework, we can neither construct the quantum algebras $U_q(\mathrm{so}_{n,1})$ nor introduce Gel'fand–Tsetlin bases for representation spaces for $U_q(\mathrm{so}_n)$. To remove these defects, the new q-deformation of the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$ (and its 'compact' real form $U'_q(\mathrm{so}_n)$) was defined by Gavrilik *et al* (1990) (see also Gavrilik and Klimyk 1991, 1994) and we denote it by $U'_q(\mathrm{so}(n,\mathbb{C}))$. This q-deformed algebra allows the embedding $U'_q(\mathrm{so}(n-1,\mathbb{C})) \subset U'_q(\mathrm{so}(n,\mathbb{C}))$ and, therefore, we can introduce Gel'fand–Tsetlin bases. It was shown by Noumi *et al* (1994) that this algebra can be embedded into $U_q(\mathrm{sl}(n,\mathbb{C}))$. This last fact makes the algebra $U'_q(\mathrm{so}_n)$ very attractive since the pairs $U'_q(\mathrm{so}_n) \subset U_q(\mathrm{u}_n)$ can be of great physical importance. Of course, for applications, we must have irreducible representations of $U'_q(\mathrm{so}_n)$ or, at least, the simplest of them. It was discovered later that the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$, defined by Gavrilik *et al* (1990), at $n = 3$, coincides with the algebra defined by Fairlie (1990). Representations of $U'_q(\mathrm{so}_3)$ were studied by Fairlie (1990). Representations of $U'_q(\mathrm{so}_4)$ with respect to the Gel'fand–Tsetlin basis were given by Gavrilik (1993). Here, we continue the study into these representations.

The algebra $U'_q(\mathrm{so}_4)$ has three generators I_{21}, I_{32} and I_{43}. Its irreducible representations T_{rs} are given by two integral or half-integral numbers r and s such that $r \geqslant |s| \geqslant 0$. We find

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the spectra and eigenvectors of the operators $T_{rs}(I_{43})$. Eigenvectors are expressed by means of q-Racah polynomials (Gasper and Rahman 1991). By making use of the eigenvectors of $T_{rs}(I_{43})$, we derive formulae for the operators $T_{rs}(I_{32})$ and $T_{rs}(I_{43})$ in the basis corresponding to the restriction of T_{rs} on the subalgebra $U'_q(\mathrm{so}_2) + U'_q(\mathrm{so}_2)$. This basis differs from the Gel'fand–Tsetlin basis. We have coefficients (overlap functions) connecting these two bases. Finally, using the explicit form of the operator $T_{rs}(I_{32})$, with respect to the $U'_q(\mathrm{so}_2)+U'_q(\mathrm{so}_2)$ basis, and by analytical continuation in parameters determining representations, we find infinite-dimensional representations of the 'non-compact' q-deformed algebra $U'_q(\mathrm{so}_{2,2})$. They are given by two complex numbers. Throughout this paper, we assume that q is not a root of unity.

2. A generalization of Fairlie's algebra

Drinfeld (1985) and Jimbo (1985) defined q-deformed (quantum) algebras $U_q(g)$ for all simple complex Lie algebras g by means of Cartan subalgebras and root subspaces. However, these approaches do not give a satisfactory presentation of the quantum algebra $U_q(\mathrm{so}(n,\mathbb{C}))$ from the point of view of some problems of quantum physics and representation theory. In fact, they admit the inclusion

$$U_q(\mathrm{so}(n,\mathbb{C})) \supset U_q(\mathrm{so}(n-2,\mathbb{C}))$$

and do not admit

$$U_q(\mathrm{so}(n,\mathbb{C})) \supset U_q(\mathrm{so}(n-1,\mathbb{C})). \tag{1}$$

This is why we cannot construct the quantum algebra $U_q(\mathrm{so}_{n,1})$ in the framework of these approaches or Gel'fand–Tsetlin bases in the representation spaces. In order to obtain inclusion (1), we proposed (Gavrilik and Klimyk 1991) another q-deformation of the classical universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$. The classical algebra $U(\mathrm{so}(n,\mathbb{C}))$ is generated by the elements $I_{i,i-1}$, $i = 2, 3, \ldots, n$ that satisfy the relations

$$I_{i,i-1}I_{i+1,i}^2 - 2I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1} \tag{2}$$

$$I_{i,i-1}^2 I_{i+1,i} - 2I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i} \tag{3}$$

$$[I_{i,i-1}, I_{j,j-1}] = 0 \qquad |i-j| > 1. \tag{4}$$

They follow from the well known commutation relations for the generators I_{ij} of the Lie algebra $\mathrm{so}(n,\mathbb{C})$ (Gel'fand and Tsetlin 1950).

In our approach to the q-deformed orthogonal algebra, we define a q-deformation of the associative algebra $U(\mathrm{so}(n,\mathbb{C}))$ by deforming relations (2)–(4). The q-deformed relations are of the form

$$I_{i,i-1}I_{i+1,i}^2 - aI_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1} \tag{5}$$

$$I_{i,i-1}^2 I_{i+1,i} - aI_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i} \tag{6}$$

$$[I_{i,i-1}, I_{j,j-1}] = 0 \qquad |i-j| > 1 \tag{7}$$

where

$$a = q^{1/2} + q^{-1/2} = (q - q^{-1})/(q^{1/2} - q^{-1/2})$$

and $[\cdot,\cdot]$ denotes the usual commutator. Obviously, in the limit $q \to 1$, formulae (5)–(7) give relations (2)–(4). We remark that relations (5) and (6) differ from the q-deformed Serre relations in the approach of Jimbo and Drinfeld to quantum orthogonal algebras by the appearance of a non-zero right-hand side and the possibility of reduction (1). Below, by the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$, we mean the q-deformed algebra defined by formulae (5)–(7).

As in the classical case, the q-algebras $U'_q(\mathrm{so}(3,\mathbb{C}))$ and $U'_q(\mathrm{so}(4,\mathbb{C}))$ can also be described in terms of bilinear relations (q-commutators). In fact, defining the algebra $U'_q(\mathrm{so}(3,\mathbb{C}))$ by relations (5)–(7) results in only two generators I_{21} and I_{32}. However, we can define the third element I_{31} according to the formula (Gavrilik and Klimyk 1994)

$$I_{31} = q^{1/4} I_{21} I_{32} - q^{-1/4} I_{32} I_{21}. \tag{8}$$

Then, by the algebra $U'_q(\mathrm{so}(3,\mathbb{C}))$, we mean the associative algebra generated by the elements I_{21}, I_{32} and I_{31}, which satisfy the relations

$$q^{1/4} I_{21} I_{32} - q^{-1/4} I_{32} I_{21} = I_{31} \tag{9}$$

$$q^{1/4} I_{31} I_{21} - q^{-1/4} I_{21} I_{31} = I_{32} \tag{10}$$

$$q^{1/4} I_{32} I_{31} - q^{-1/4} I_{31} I_{32} = I_{21}. \tag{11}$$

It is clear that if the generators I_{21}, I_{32} and I_{31} satisfy relations (9)–(11), then the pair I_{21} and I_{32} satisfy the trilinear relations (5) and (6). We remark that the algebra given by relations (9)–(11) coincides with the cyclically symmetric Fairlie algebra (Fairlie 1990). For this reason, we call our q-deformed algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ a generalization of the Fairlie algebra. It was shown by Noumi *et al* (1994) that this algebra can be embedded into the Drinfeld–Jimbo algebra $U_q(\mathrm{sl}(n,\mathbb{C}))$. In particular, we have the embedding $U'_q(\mathrm{so}(3,\mathbb{C})) \subset U_q(\mathrm{sl}(3,\mathbb{C}))$ which is important from the point of view of nuclear physics. It was shown by Noumi (1994) that the algebra $U'_q(\mathrm{so}(n,\mathbb{C}))$ allows us to define quantum analogues of the homogeneous spaces $GL(n)/SO(n)$.

The q-deformed algebra $U'_q(\mathrm{so}(4,\mathbb{C}))$ is generated by I_{21}, I_{32} and I_{43}. Moreover, for the first two generators, everything, concerning $U'_q(\mathrm{so}(3,\mathbb{C}))$ above, is true. Thus, the inclusion

$$U'_q(\mathrm{so}(3,\mathbb{C})) \subset U'_q(\mathrm{so}(4,\mathbb{C}))$$

takes place. The generators I_{21} and I_{43} mutually commute (see relation (7)) and the pair I_{32}, I_{43} must, in turn, satisfy relations (5) and (6). Again, $U'_q(\mathrm{so}(4,\mathbb{C}))$ can also be given in terms of bilinear q-commutators. Namely, we can add the element I_{31} from (8) and the elements I_{42} and I_{41} defined as

$$I_{42} = q^{1/4} I_{32} I_{43} - q^{-1/4} I_{43} I_{32} \tag{12}$$

$$I_{41} = q^{1/4} I_{31} I_{43} - q^{-1/4} I_{43} I_{31} = q^{1/4} I_{21} I_{42} - q^{-1/4} I_{42} I_{21} \tag{13}$$

to the triplet of generators I_{21}, I_{32} and I_{43}.

Various real forms of our algebras $U_q'(\mathrm{so}(n,\mathbb{C}))$ are obtained by introducing corresponding $*$-structures (antilinear antiautomorphisms). The compact real form $U_q'(\mathrm{so}_n)$ is defined by the $*$-structure

$$I_{i,i-1}^* = -I_{i,i-1} \qquad i = 2, 3, \ldots, n. \tag{14}$$

The non-compact quantum algebras $U_q'(\mathrm{so}_{p,r})$, where $r = n - p$, are distinguished respectively by means of the $*$-structures ($i \leqslant n$)

$$I_{i,i-1}^* = -I_{i,i-1} \qquad i \neq p+1 \qquad I_{p+1,p}^* = I_{p+1,p}. \tag{15}$$

In this paper, we consider finite-dimensional representations of the algebra $U_q'(\mathrm{so}_4)$ (they are also representations of $U_q'(\mathrm{so}(4,\mathbb{C}))$) and infinite-dimensional representations of the non-compact algebra $U_q'(\mathrm{so}_{2,2})$.

3. Representations of Fairlie's algebra

Since Fairlie's algebra $U_q'(\mathrm{so}_3)$ is a subalgebra of $U_q'(\mathrm{so}_4)$, we need finite-dimensional irreducible representations of $U_q'(\mathrm{so}_3)$. As shown by Fairlie (1990), these representations are given by integral or half-integral non-negative numbers l. We denote these representations by T_l. The carrier space of the representation T_l has the orthonormal basis $\{|m\rangle, m = l, l-1, \ldots, -l\}$ and the operators $T_l(I_{21})$ and $T_l(I_{32})$ act upon this basis as

$$T_l(I_{21})|m\rangle = \mathrm{i}[m]|m\rangle \tag{16}$$

$$T_l(I_{32})|m\rangle = d(m)([l-m][l+m+1])^{1/2}|m+1\rangle - d(m-1)([l-m+1][l+m])^{1/2}|m-1\rangle \tag{17}$$

where

$$d(m) = ([m][m+1]/[2m][2m+2])^{1/2}$$

and $[a]$ denotes a q-number defined by the formula

$$[a] = (q^{a/2} - q^{-a/2})/(q^{1/2} - q^{-1/2}).$$

For the operator $T_l(I_{31})$, we have

$$T_l(I_{31})|m\rangle = \mathrm{i}q^{1/4}\{q^{m/2}d(m)([l-m][l+m+1])^{1/2}|m+1\rangle + q^{-m/2}d(m-1)([l-m+1][l+m])^{1/2}|m-1\rangle\}. \tag{18}$$

Let us note that the operators $T_l(I_{21})$ and $T_l(I_{32})$ satisfy conditions (14). The operator $T_l(I_{31})$ is not anti-Hermitian since, in the algebra $U_q'(\mathrm{so}_3)$, we have $I_{31}^* \neq -I_{31}$.

4. Representations of the algebra $U_q'(\mathrm{so}_4)$ in the Gel'fand–Tsetlin basis

As in the case of the Lie group $SO(4)$, finite-dimensional irreducible representations T_{rs} of the q-deformed algebra $U_q'(\mathrm{so}_4)$ are given by two integral or half-integral numbers r and s such that $r \geqslant |s| \geqslant 0$ (Gavrilik 1993). Restriction of T_{rs} on subalgebra $U_q'(\mathrm{so}_3)$ decomposes into the sum of the irreducible representations T_l of this subalgebra for which $l = |s|, |s|+1, \ldots, r$. Uniting the bases of the subspaces of the irreducible representations T_l of $U_q'(\mathrm{so}_3)$, we obtain the basis of the carrier space V_{rs} of the representation T_{rs} of $U_q'(\mathrm{so}_4)$. Thus, the corresponding orthonormal basis of V_{rs} consists of the vectors

$$|l, m\rangle \qquad |s| \leqslant l \leqslant r \qquad m = -l, -l+1, \ldots, l.$$

The operator $T_{rs}(I_{43})$ acts upon these vectors by the formula (Gavrilik 1993)

$$\begin{aligned} T_{rs}(I_{43})|l, m\rangle = {} & \mathrm{i}\frac{[r+1][s][m]}{[l][l+1]}|l, m\rangle \\ & + \left(\frac{[r-l][l+s+1][l-s+1][l+m+1][l-m+1]}{[r+l+2]^{-1}[l+1]^2[2l+1][2l+3]}\right)^{1/2} |l+1, m\rangle \\ & - \left(\frac{[r+l+1][l+s][l-s][l+m][l-m]}{[r-l+1][l]^2[2l-1][2l+1]}\right)^{1/2} |l-1, m\rangle \end{aligned} \tag{19}$$

where numbers in square brackets are q-numbers. The operators $T_{rs}(I_{21})$ and $T_{rs}(I_{32})$ act upon the basis vectors by formulae (16) and (17). Formulae (16), (17) and (19) completely determine the representation T_{rs}.

5. Diagonalization of the operator $T_{rs}(I_{43})$

Here we diagonalize the operator $T_{rs}(I_{43})$ in order to use this result in the next section for obtaining representations T_{rs} in the bases corresponding to restriction upon subalgebra $U_q'(\mathrm{so}_2)+U_q'(\mathrm{so}_2)$. It is more convenient to deal with the operator $L = -\mathrm{i}T_{rs}(I_{43})$ ($\mathrm{i} = \sqrt{-1}$) as it is self-adjoint. Replacing the vectors $|l, m\rangle$ by $|l, m\rangle' = \mathrm{i}^{-l}|l, m\rangle$, we obtain that L acts upon the vectors $|l, m\rangle'$ by formula (19) in which the sign $-$ of the third summand is replaced by $+$ and the first summand is multipled by $-\mathrm{i}$.

The space V_{rs} can be decomposed into the sum

$$V_{rs} = \sum_{m=-s}^{r} \oplus V_m$$

where V_m is spanned by the vectors $|l, m\rangle$ with fixed m. Let us find the spectrum and the eigenvectors

$$|x, m\rangle' = \sum_{l=k}^{r} P_{l-k}(x)|l, m\rangle \qquad k = \max(|m|, |s|) \tag{20}$$

of the operator L on the subspace V_m:

$$L|x, m\rangle' = [x]|x, m\rangle' \tag{21}$$

where $[x]$ is a q-number. Formula (19) is symmetric with respect to permutation of s and m and to sign changes at m and s. Therefore, we may assume, without loss of generality, that s and m are positive and that $s \geqslant m$.

Substituting expression (20) for $|x, m\rangle'$ into (21) and acting on $|l, m\rangle$ by L, we find easily that vector (20) is an eigenvector of L with eigenvalue $[x]$ if P_{l-k} satisfies the recurrence relation

$$\left(\frac{[u][n+2s+1][n+1][n+s+m+1][n+s-m+1]}{[r+n+s+2]^{-1}[n+s+1]^2[2n+2s+1][2n+2s+3]}\right)^{1/2} P_{n+1}(x) + \left(\frac{[r+n+s+1][r-n-s+1][n+2s][n][n+s+m]}{[n+s-m]^{-1}[n+s]^2[2n+2s-1][2n+2s+1]}\right)^{1/2} P_{n-1}(x) + \frac{[r+1][s][m]}{[n+s][n+s+1]} P_n(x) = [x]P_n(x) \tag{22}$$

(here, $u = r-n-s$, $n = l-k$) and the initial conditions $P_0(x) = 1$, $P_{-1}(x) = 0$.

Making, in (22), the substitution

$$P_n(x) = -q^c \left(\frac{[n+2s]![n+s+m]![2n+2s+1]}{[n]![n+s-m]![r-n-s]![r+n+s+1]!}\right)^{1/2} P'_n(x)$$

where $c = (s+m-r)/2$, we reduce (22) to the recurrence relation

$$\frac{(1-q^{n+2s+1})(1-q^{n+s+m+1})(1-q^{n-r+s})(1+q^{n+s+1})}{(1-q^{2n+2s+1})(1-q^{2n+2s+2})} P'_{n+1}(x) - \frac{q^{s+m-r}(1-q^n)(1-q^{r+n+s+1})(1+q^{n+s})(1-q^{n+s-m})}{(1-q^{2n+2s+1})(1-q^{2n+2s})} P'_{n-1}(x) - \frac{q^{n-r+s}(1-q^{r+1})(1-q^m)(1-q^s)}{(1-q^{n+s})(1-q^{n+s+1})} P'_n(x) = \frac{q^{1/2}-q^{-1/2}}{q^{(r-s-m)/2}}[x]P'_n(x).$$

Comparing this formula with recurrence relation (7.5.2), from the book by Gasper and Rahman (1991) for q-Racah polynomials,

$$R_n(\mu(y); \alpha, \beta, \gamma, \delta|q) = {}_4\varphi_3\left(\begin{matrix} q^{-y}, q^{y+1}\gamma\delta, q^{-n}, q^{n+1}\alpha\beta \\ \alpha q, \beta\delta q, \gamma q \end{matrix}; q, q\right)$$

(here, ${}_4\varphi_3$ is a basic hypergeometric function which can be found in Gasper and Rahman (1991)) at

$$\alpha = \beta = -q^s \qquad \gamma = q^{s+m} \qquad \delta = -q^{-r-1}. \tag{23}$$

After cumbersome transformations, we conclude that

$$P'_n(x) = R_n(\mu(y);\ \alpha, \beta, \gamma, \delta|q)$$

where α, β, γ and δ are given by formulae (23) and

$$x = (r-s-m) - 2y.$$

Thus, the polynomials $P_n(x)$ from (22), normalized by the condition $P_0(x) = 1$, are of the form

$$P_n(x) = N^{1/2} R_n(\mu(y);\ -q^s, -q^s, q^{s+m}, -q^{-r-1}|q) \tag{24}$$

$$N = \frac{[n+2s]![n+s+m]![2n+2s+1][s-m]![r-s]![r+s+1]!}{[n]![n+s-m]![r-n-s]![r+n+s+1]![2s]![s+m]![2s+1]}$$

where $x = (r-s-m) - 2y$. The variable y takes the values $0, 1, 2, \ldots, r-s$. Therefore, the spectrum of L on the subspace V_m consists of the points

$$[r-s-m], [r-s-2-m], [r-s-4-m], \ldots, [-(r-s)-m]. \tag{25}$$

The corresponding eigenvectors are determined by formulae (20) and (24). The orthogonality relation for the polynomials $P_n(x)$ follows from the orthogonality of q-Racah polynomials (Gasper and Rahman 1991) and is of the form

$$\sum_{y=0}^{r-s} P_n(x) P_k(x) W(x) = \delta_{nk}. \tag{26}$$

Here, $W(x)$ is equal to the expression

$$\frac{[4y+2k-2r][2y+2k-2r-2]!![2y+2s]!![2r-2y]!![r-m-y]!}{[2y+2k-2r][y+k-r-1]![y+s]![2y+2m]!![r-y]![r-s-y]![y]!}$$
$$\times [y+m]![k+y]![2s+1]!!([s]!)^2[r-s]([2s]!![s-m]![k]![r+s+1]!)^{-1}$$

where $k = s+m$, $[n]! = [n][n-1]\cdots[1]$ and $[n]!! = [n][n-2][n-4]\cdots[1]$ or $[2]$.

Formula (26) shows that vectors (20) are not normalized. The vectors

$$|x, m\rangle = W(x)^{1/2}|x, m\rangle'$$

are normal and, due to formula (21), we have

$$T_{rs}(I_{43})|x, m\rangle = \mathrm{i}[x]|x, m\rangle. \tag{27}$$

Joining spectra (25) for all subspaces V_m, we obtain the spectrum of the operator T_{rs} and, therefore, also of the operator $T_{rs}(I_{43})$.

6. Representations T_{rs} in the basis $|x, m\rangle$

The operator $T_{rs}(I_{43})$ acts upon the basis vectors $|x, m\rangle$ by formula (27). It is clear from formulae (16) and (20) that

$$T_{rs}(I_{21})|x, m\rangle = \mathrm{i}[m]|x, m\rangle. \tag{28}$$

Thus, to have representation T_{rs} in the basis $|x, m\rangle$, we must find the action formula for the operator $T_{rs}(I_{32})$ upon this basis.

Since

$$|x, m\rangle = \sum_{l=s}^{r} P_{l-s}^{m}(x)|l, m\rangle \tag{29}$$

with $P_{l-s}^{m}(x) = W(x)^{1/2}P_{l-s}(x)$, then due to formula (17) we have

$$T_{rs}(I_{32})|x, m\rangle = d(m)\sum_{l=s}^{r} P_{l-s}^{m}(x)([l-m][l+m+1])^{1/2}|l, m+1\rangle$$
$$- d(m-1)\sum_{l=s}^{r} P_{l-s}^{m}(x)([l-m+1][l+m])^{1/2}|l, m-1\rangle. \tag{30}$$

Applying recurrence relation (7.2.14) of Gasper and Rahman (1991) with

$$a = q^{m-r-1} \qquad b = -q^{s} \qquad c = d = -q^{m}$$
$$n = (r-s-m-x)/2 \qquad j = l-m$$

to $([l-m][l+m+1])^{1/2}P_{l-s}^{m}(x)$ and using the equalities

$$[2x]/[x] = q^{x/2} + q^{-x/2} \qquad (q^{(a+b)/2} \pm q^{-(a+b)/2})(q^{(a-b)/2} \mp q^{-(a-b)/2}) = [2a] \mp [2b]$$

after some calculations we obtain for the first summand of the right-hand side of (30) the expression

$$d(m)d(x-1)\{([r+1]+[s-m+x-1])([r+1]+[s+m-x+1])\}^{1/2}\sum_{l=s}^{r} P_{l-s}^{m+1}(x-1)|l, m+1\rangle$$
$$- d(m)d(x)\{([r+1]-[s+m+x+1])([r+1]-[s-m-x-1])\}^{1/2}$$
$$\times \sum_{l=s}^{r} P_{l-s}^{m+1}(x+1)|l, m+1\rangle. \tag{31}$$

To transform the second summand on the right-hand side of (30), we apply the transformation

$${}_4\varphi_3\left(\begin{matrix} q^{-N}, \alpha, \beta, \gamma \\ \delta, \sigma, \rho \end{matrix}; q, q\right) = \frac{(\sigma/\alpha; q)_N(\rho/\alpha; q)_N}{(\sigma; q)_N(\rho; q)_N}\alpha^N \, {}_4\varphi_3\left(\begin{matrix} q^{-N}, \alpha, \delta/\beta, \delta/\gamma \\ \delta, \alpha q^{1-N}/\sigma, \alpha q^{1-N}/\rho \end{matrix}; q, q\right)$$

to the basic hypergeometric function ${}_4\varphi_3$ from the expression for $P_{l-s}^{m}(x)$ (see Gasper and Rahman 1991), where $N = l-s$ and

$$\alpha = q^{l+s+1} \qquad \beta = -q^{(s-r+m-x)/2} \qquad \gamma = q^{(s-r+m+x)/2}$$
$$\delta = q^{s-r} \qquad \sigma = -q^{s+1} \qquad \rho = q^{s+m+1}.$$

Here $(a; q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$. Now, we apply the same recurence relation (7.2.14) of Gasper and Rahman (1991) with

$$a = q^{-(r+m+1)} \qquad b = -q^{-s} \qquad c = d = -q^{m}$$
$$n = (r-s+m+x)/2 \qquad j = l+m$$

to $([l-m+1][l+m])^{1/2}P^m_{l-s}(x)$.

Then, the second summand of the right-hand side of (30) takes the form

$$d(m-1)d(x)\{([r+1]+[s+m-x-1])([r+1]+[s-m+x+1])\}^{1/2}$$
$$\times\sum_{l=s}^{r}P^{m-1}_{l-s}(x+1)|l,m-1\rangle$$
$$-d(m-1)d(x-1)\{([r+1]-[s-m-x+1])([r+1]-[s+m+x-1])\}^{1/2}$$
$$\times\sum_{l=s}^{r}P^{m-1}_{l-s}(x-1)|l,m-1\rangle. \tag{32}$$

We substitute expressions (31) and (32) into (30) and take into account formula (29). As a result, we find that the operator $T_{rs}(I_{32})$ acts upon the vectors $|x,m\rangle$ as

$$T_{rs}(I_{32})|x,m\rangle = d(m)d(x-1)\{([r+1]+[s-m+x-1])([r+1]+[s+m-x+1])\}^{1/2}$$
$$\times|x-1,m+1\rangle$$
$$-d(m)d(x)\{([r+1]-[s+m+x+1])([r+1]-[s-m-x-1])\}^{1/2}$$
$$\times|x+1,m+1\rangle$$
$$+d(m-1)d(x-1)\{([r+1]-[s-m-x+1])([r+1]-[s+m+x-1])\}^{1/2}$$
$$\times|x-1,m-1\rangle$$
$$-d(m-1)d(x)\{([r+1]+[s+m-x-1])([r+1]+[s-m+x+1])\}^{1/2}$$
$$\times|x+1,m-1\rangle. \tag{33}$$

Now, we have completely determined the representations T_{rs} of $U'_q(\mathrm{so}_4)$ with respect to the basis corresponding to reduction onto the subalgebra $U'_q(\mathrm{so}_2)+U'_q(\mathrm{so}_2)$.

7. Representations of the q-deformed algebra $U'_q(\mathrm{so}_{2,2})$

As in the case of representations of compact and non-compact real Lie groups, by making use of analytical continuation in parameters giving representations, we can obtain infinite-dimensional representations of the q-deformed algebra $U'_q(\mathrm{so}_{2,2})$ from the representations T_{rs} of $U'_q(\mathrm{so}_4)$. In this way, we obtain the representations $T^{\epsilon}_{\sigma\tau}$, $\sigma\in\mathbb{C}$, $\tau\in\mathbb{C}$, $\epsilon\in\{0,1\}$ of $U'_q(\mathrm{so}_{2,2})$ which act on the Hilbert spaces H_ϵ with the orthonormal basis

$$|x,m\rangle \qquad x\in\mathbb{Z} \qquad m\in\mathbb{Z} \qquad x+m\equiv\epsilon(\mathrm{mod}\,2).$$

The operators $T^{\epsilon}_{\sigma\tau}(I_{21})$ and $T^{\epsilon}_{\sigma\tau}(I_{43})$ act upon these basis vectors by formulae (27) and (28). For the operator $T^{\epsilon}_{\sigma\tau}(I_{32})$, we have

$$T^{\epsilon}_{\sigma\tau}(I_{32})|x,m\rangle = d(m)d(x-1)\{([\sigma+1]+[\tau-m+x-1])([\sigma+1]+[\tau+m-x+1])\}^{1/2}$$
$$\times|x-1,m+1\rangle - d(m)d(x)$$
$$\times\{([\sigma+1]-[\tau+m+x+1])([\sigma+1]-[\tau-m-x-1])\}^{1/2}$$
$$\times|x+1,m+1\rangle + d(m-1)d(x-1)$$
$$\times\{([\sigma+1]-[\tau-m-x+1])([\sigma+1]-[\tau+m+x-1])\}^{1/2}$$
$$\times|x-1,m-1\rangle - d(m-1)d(x)$$
$$\times\{([\sigma+1]+[\tau+m-x-1])([\sigma+1]+[\tau-m+x+1])\}^{1/2}$$
$$\times|x+1,m-1\rangle. \tag{34}$$

To analyse the irreducibility of these representations, it is more convenient to transform formula (34) into the following form:

$$
\begin{aligned}
T^{\epsilon}_{\sigma\tau}(I_{32})|x,m\rangle = & \left(\frac{[\sigma-\tau+m-x+2][\sigma-\tau-m+x][(\sigma+\tau+m-x+2)/2]}{[(\sigma-\tau+m-x+2)/2][(\sigma-\tau-m+x)/2][(\sigma+\tau-m+x)/2]^{-1}}\right)^{1/2} \\
& \times d(m)d(x-1)|x-1,m+1\rangle \\
& -\left(\frac{[\sigma+\tau+m+x+2][\sigma+\tau-m-x][(\sigma-\tau+m+x+2)/2]}{[(\sigma+\tau+m+x+2)/2][(\sigma+\tau-m-x)/2][(\sigma-\tau-m-x)/2]^{-1}}\right)^{1/2} \\
& \times d(m)d(x)|x+1,m+1\rangle \\
& +\left(\frac{[\sigma+\tau-m-x+2][\sigma+\tau+m+x][(\sigma-\tau-m-x+2)/2]}{[(\sigma+\tau-m-x+2)/2][(\sigma+\tau+m+x)/2][(\sigma-\tau+m+x)/2]^{-1}}\right)^{1/2} \\
& \times d(m-1)d(x-1)|x-1,m-1\rangle \\
& -\left(\frac{[\sigma-\tau-m+x+2][\sigma-\tau+m-x][(\sigma+\tau-m+x+2)/2]}{[(\sigma-\tau-m+x+2)/2][(\sigma-\tau+m-x)/2][(\sigma+\tau+m-x)/2]^{-1}}\right)^{1/2} \\
& \times d(m-1)d(x)|x+1,m-1\rangle. \qquad (35)
\end{aligned}
$$

In every summand here, there are two expressions of the form

$$[\sigma-\tau-m+x]/[(\sigma-\tau-m+x)/2].$$

This expression is equal to

$$q^{(\sigma-\tau-m+x)/4}+q^{-(\sigma-\tau-m+x)/4}.$$

Irreducibility of the representations $T^{\epsilon}_{\sigma\tau}$ is studied in the same way as in the case of the q-deformed algebras $U'_q(\mathrm{so}_{2,1})$ and $U'_q(\mathrm{so}_{3,1})$ (Gavrilik and Klimyk 1994). Namely, invariant subspaces in the representation space appear because of the vanishing of some coefficients in formula (35). This studying leads to the following result:

Theorem 1. A representation $T^{\epsilon}_{\sigma\tau}$ of the algebra $U'_q(\mathrm{so}_{2,2})$ is irreducible if and only if $\sigma+\tau \not\equiv \epsilon \pmod 2$ and $\sigma-\tau \not\equiv \epsilon \pmod 2$.

The reducible representations $T^{\epsilon}_{\sigma\tau}$ will be studied in a forthcoming paper.

8. Conclusion

We obtained spectra of the operators $T_{rs}(I_{43})$ of irreducible representations of the q-deformed algebra $U'_q(\mathrm{so}_4)$. In contrast to the classical case, spectra are not equidistant or simple. Multiplicities of spectral points are calculated with the help of irreducible representations T_l of the subalgebra $U'_q(\mathrm{so}_3)$.

By making use of the spectra of the operators $T_{rs}(I_{43})$, we derived an explicit formula for the operator $T_{rs}(I_{32})$ with respect to the basis $|x,m\rangle$, corresponding to restriction onto the subalgebra $U'_q(\mathrm{so}_2)+U'_q(\mathrm{so}_2)$. The other generating operators $T_{rs}(I_{21})$ and $T_{rs}(I_{43})$ are diagonal in this basis. Formulae for the operators $T_{rs}(I_{i,i-1})$, $i=2,3,4$, determine the representation T_{rs} in the basis $|x,m\rangle$, which differs from the Gel'fand–Tsetlin basis. Analytical continuation in parameters determining representations of $U'_q(\mathrm{so}_4)$ leads to irreducible representations of the q-deformed algebra $U'_q(\mathrm{so}_{2,2})$ characterized by two complex numbers.

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